

# Diffusion time and splitting of separatrices for nearly integrable isochronous Hamiltonian systems

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**Abstract:** We consider the problem of Arnold's diffusion for nearly integrable isochronous Hamiltonian systems. We prove a shadowing theorem which improves the known estimates for the diffusion time. We also justify for three time scales systems that the splitting of the separatrices is correctly predicted by the Poincaré-Melnikov function.<sup>1</sup>

Keywords: Arnold's diffusion, shadowing, splitting of separatrices, heteroclinic orbits, variational methods.

**Riassunto:** *Tempo di diffusione e splitting delle separatrici per sistemi Hamiltoniani isocroni quasi-integrabili.* Consideriamo il problema della diffusione di Arnold per sistemi Hamiltoniani isocroni quasi-integrabili. Dimostriamo un teorema di shadowing che migliora le stime sul tempo di diffusione sinora note. Giustificiamo inoltre, per sistemi a tre scale temporali, che lo splitting delle separatrici è correttamente previsto dalla funzione di Poincaré-Melnikov.

## 1 Introduction

We outline in this Note some recent results on Arnold's diffusion for nearly integrable isochronous Hamiltonian systems: complete proofs and further results are contained in [6]. We consider Hamiltonians  $\mathcal{H}_\mu$  of the form

$$\mathcal{H}_\mu = \omega \cdot I + \frac{p^2}{2} + (\cos q - 1) + \mu f(\varphi, q) \quad (1.1)$$

with angles variables  $(\varphi, q) \in \mathbf{T}^n \times \mathbf{T}^1$  and action variables  $(I, p) \in \mathbf{R}^n \times \mathbf{R}^1$ . We assume

- (H1) There exists  $\gamma > 0$ ,  $\tau > n$  such that  $|\omega \cdot k| \geq \gamma/|k|^\tau$ ,  $\forall k \in \mathbf{Z}^n, k \neq 0$ .

Hamiltonian  $\mathcal{H}_\mu$  describes a system of  $n$  isochronous harmonic oscillators of frequencies  $\omega$  weakly coupled with a pendulum. When  $\mu = 0$  the energy  $\omega_i I_i$  of each oscillator is a constant of the motion. The problem of *Arnold's diffusion* is whether, for  $\mu \neq 0$ , there exists motions whose net effect is to transfer energy from one oscillator to the others.

Stemming from [3] these kind of results are usually proved noting that, for  $\mu = 0$  Hamiltonian  $\mathcal{H}_\mu$  admits a continuous family of  $n$ -dimensional partially hyperbolic invariant tori  $\mathcal{T}_{I_0} = \{(\varphi, I, q, p) \in \mathbf{T}^n \times \mathbf{R}^n \times \mathbf{T}^1 \times \mathbf{R}^1 \mid I = I_0, q = p = 0\}$  possessing stable and unstable manifolds  $W^s(\mathcal{T}_{I_0}) = W^u(\mathcal{T}_{I_0}) = \{(\varphi, I, q, p) \in \mathbf{T}^n \times \mathbf{R}^n \times \mathbf{T}^1 \times \mathbf{R}^1 \mid I = I_0, p^2/2 + (\cos q - 1) = 0\}$ . By (H1) all the unperturbed tori  $\mathcal{T}_{I_0}$ , with their stable and unstable manifolds, persist, for  $\mu$  small enough, being just slightly deformed. The perturbed stable and unstable manifolds  $W_\mu^s(\mathcal{T}_{I_0}^\mu)$  and  $W_\mu^u(\mathcal{T}_{I_0}^\mu)$  may split and intersect transversally giving rise to a chain of tori connected by heteroclinic orbits. By a shadowing type argument one can then prove the existence of an orbit such that the action variables  $I$  undergo a variation of  $O(1)$  in a certain time  $T_d$  called the *diffusion time*. In order to prove the existence of diffusion orbits following the previous mechanism one encounters two different problems

- 1) Shadowing theorem;
- 2) Splitting of separatrices;

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By means of a variational technique inspired from [4]-[5]-[1] we give in the next section a general shadowing theorem which improves, for isochronous systems, the known estimates on the diffusion time obtained in [8]-[11]-[16]-[9] by geometric methods and in [7] by Mather's theory. In section 3, using methods introduced in [2], we give some results on the splitting of the separatrices.

## 2 The shadowing theorem

### 2.1 Perturbation term vanishing along the invariant tori

We first describe our results when the perturbation term is  $f(\varphi, q) = (1 - \cos q) f(\varphi)$  so that the tori  $\mathcal{T}_{I_0}$  are still invariant for  $\mu \neq 0$ . The equations of motion derived by Hamiltonian  $\mathcal{H}_\mu$  are

$$\dot{\varphi} = \omega, \quad \dot{I} = -\mu(1 - \cos q) \partial_\varphi f(\varphi), \quad \dot{q} = p, \quad \dot{p} = \sin q - \mu \sin q f(\varphi). \quad (2.1)$$

The angles  $\varphi$  evolve as  $\varphi(t) = \omega t + A$  and then (2.1) can be reduced to the quasi-periodically forced pendulum equation

$$-\ddot{q} + \sin q = \mu \sin q f(\omega t + A), \quad (2.2)$$

corresponding to the Lagrangian

$$\mathcal{L}_\mu(q, \dot{q}, t) = \frac{\dot{q}^2}{2} + (1 - \cos q) + \mu(\cos q - 1)f(\omega t + A). \quad (2.3)$$

For each solution  $q(t)$  of (2.2) one recovers the dynamics of the actions  $I(t)$  by quadratures in (2.1).

For  $\mu = 0$  equation (2.2) possesses the family of heteroclinic solutions  $q_\theta(t) = 4 \arctg(\exp(t - \theta))$ ,  $\theta \in \mathbf{R}$ . Using the Contraction Mapping Theorem as in [4] one can prove that near the unperturbed heteroclinic solutions  $q_\theta(t)$  there exist, for  $\mu$  small enough, “pseudo-heteroclinic solutions”  $q_{A,\theta}^\mu(t)$  of equation (2.2).  $q_{A,\theta}^\mu(t)$  are true solutions of (2.2) in each interval  $(-\infty, \theta)$  and  $(\theta, +\infty)$ ; at time  $t = \theta$  such pseudo-solutions are glued with continuity at value  $q_{A,\theta}^\mu(\theta) = \pi$  and for  $t \rightarrow \pm\infty$  are asymptotic to the equilibrium 0 mod  $2\pi$ . Moreover, by a uniqueness property,  $q_{A,\theta}^\mu$  depends smoothly on  $(\mu, A, \theta)$ . We can then define the function  $F_\mu : \mathbf{T}^n \times \mathbf{R} \rightarrow \mathbf{R}$  as the action functional of Lagrangian (2.3) evaluated on the “1-bump pseudo-heteroclinic solutions”  $q_{A,\theta}^\mu(t)$ , namely

$$F_\mu(A, \theta) = \int_{-\infty}^{\theta} \mathcal{L}_\mu(q_{A,\theta}^\mu(t), \dot{q}_{A,\theta}^\mu(t), t) dt + \int_{\theta}^{+\infty} \mathcal{L}_\mu(q_{A,\theta}^\mu(t), \dot{q}_{A,\theta}^\mu(t), t) dt, \quad (2.4)$$

and the “homoclinic function”  $G_\mu : \mathbf{T}^n \rightarrow \mathbf{R}$  as

$$G_\mu(A) = F_\mu(A, 0). \quad (2.5)$$

There holds  $F_\mu(A, \theta) = G_\mu(A + \omega\theta)$ ,  $\forall \theta \in \mathbf{R}$ .

**Remark 2.1** The homoclinic function  $G_\mu$  is the difference between the generating functions  $\mathcal{S}_{\mu, I_0}^\pm(A, q)$  of the stable and the unstable manifolds  $W_\mu^{s,u}(\mathcal{T}_{I_0})$  (which in this case are exact Lagrangian manifolds) at the section  $q = \pi$ , namely  $G_\mu(A) = \mathcal{S}_{\mu, I_0}^-(A, \pi) - \mathcal{S}_{\mu, I_0}^+(A, \pi)$ . Note that  $G_\mu$  is independent of  $I_0$ .

We now give an example of condition on  $G_\mu$  which implies the existence of diffusion orbits.

**Condition 2.1** (“Splitting condition”) There exist  $A_0 \in \mathbf{T}^n$ ,  $\delta > 0$ ,  $0 < \alpha < \rho$  such that

- (i)  $\inf_{\partial B_\rho(A_0)} G_\mu > \inf_{B_\rho(A_0)} G_\mu + \delta$ ;
- (ii)  $\sup_{B_\alpha(A_0)} G_\mu < \frac{\delta}{4} + \inf_{B_\rho(A_0)} G_\mu$ ;
- (iii)  $d(\{A \in B_\rho(A_0) \mid G_\mu(A) < \delta/2 + \inf_{B_\rho(A_0)} G_\mu\}, \{A \in B_\rho(A_0) \mid G_\mu(A) \geq 3\delta/4 + \inf_{B_\rho(A_0)} G_\mu\}) \geq 2\alpha$ .

Note that the above “splitting condition” is clearly satisfied if  $G_\mu$  possesses in  $A_0 \in \mathbf{T}^n$  a non-degenerate minimum. Moreover  $B_\rho(A_0)$ , open ball of radius  $\rho$ , could be replaced by an open subset  $U$  of  $\mathbf{T}^n$  whose covering set is bounded in  $\mathbf{R}^n$ .

The following shadowing type theorem holds

**Theorem 2.1** *Assume (H1) and let  $G_\mu$  satisfy the splitting condition 2.1. Then  $\forall I_0, I'_0$  with  $\omega \cdot I_0 = \omega \cdot I'_0$ , there is a heteroclinic orbit connecting the invariant tori  $\mathcal{T}_{I_0}$  and  $\mathcal{T}_{I'_0}$ . Moreover, there exists  $C > 0$  such that  $\forall \eta > 0$  small the “diffusion time”  $T_d$  needed to go from a  $\eta$ -neighbourhood of  $\mathcal{T}_{I_0}$  to a  $\eta$ -neighbourhood of  $\mathcal{T}_{I'_0}$  is bounded by*

$$T_d \leq C \frac{|I_0 - I'_0|}{\delta} \rho \max \left( |\ln \delta|, \frac{1}{\alpha^\tau} \right) + C |\ln(\eta)|. \quad (2.6)$$

**Remark 2.2** *The meaning of (2.6) is the following: the diffusion time  $T_d$  is estimated by the product of the number of heteroclinic transitions  $k = (\text{heteroclinic jump} / \text{splitting}) = |I'_0 - I_0|/\delta$ , and of the time  $T_s$  required for a single transition, that is  $T_d = k \cdot T_s$ . The time for a single transition  $T_s$  is bounded by the maximum time between the “ergoditation time”  $(1/\alpha^\tau)$ , i.e. the time needed for the flow  $\omega t$  to make an  $\alpha$ -net of the torus, and the time  $|\ln \delta|$  needed to “shadow” homoclinic orbits for the forced pendulum equation. We use here that these homoclinic orbits are exponentially asymptotic to the equilibrium.*

When the frequency vector  $\omega$  is considered as a constant, independent of any parameter (“a-priori unstable” case), it is easy to give a criterion for the splitting condition 2.1 thanks to the first order expansion in  $\mu$ . There holds

$$G_\mu(A) = \text{const} + \mu \Gamma(A) + O(\mu^2), \quad (2.7)$$

where  $\Gamma : \mathbf{T}^n \rightarrow \mathbf{R}$  is the Poincaré-Melnikov primitive

$$\Gamma(A) = \int_{\mathbf{R}} (1 - \cos q_0(t)) f(\omega t + A) dt. \quad (2.8)$$

As a corollary of theorem 2.1 we have

**Corollary 2.1** *Assume (H1) and let  $\Gamma$  possess a non-degenerate minimum. Then, for  $\mu$  small enough, the statement of theorem 2.1 holds where the diffusion time is*

$$T_d = O\left(\frac{1}{\mu} \log \frac{1}{\mu}\right). \quad (2.9)$$

**Remark 2.3** *The estimate on the diffusion time obtained in [8] is  $T_d \gg O(\exp 1/\mu)$  and is improved in [11] to be  $T_d = O(\exp 1/\mu)$ . Recently in [7] by means of Mather’s theory the estimate on the diffusion time has been improved to be  $T_d = O(1/\mu^{2\tau+1})$ . In [9] it is obtained via geometric methods that  $T_d = O(1/\mu^{\tau+1})$ . The main reason for which we manage to improve also the estimates of [7] and [9] is that the shadowing orbit of theorem 2.1 can be chosen, at each transition, to approach the homoclinic point  $A_0$ , only up to the distance  $\alpha$  which does not depend of  $\mu$ . Note moreover that estimate (2.9) is independent of the number of rotators  $n$ .*

**Remark 2.4** *The above result answers to a question raised in [14] (sec.7) proving that, at least for isochronous systems, it is possible to reach the maximal speed of diffusion  $\mu/|\log \mu|$  (moreover independently on the dimension  $n$ ).*

## 2.2 More general perturbation term

Dealing with more general perturbations  $f(\varphi, q)$  the first step is to prove the persistence of invariant tori for  $\mu \neq 0$  small enough. It appears that no more than the standard Implicit Function Theorem is required to prove the following well known result.

**Theorem 2.2** *Let  $\omega$  satisfy (H1). For  $\mu$  small enough, for all  $I_0 \in \mathbf{R}^n$  system  $\mathcal{H}_\mu$  possesses  $n$ -dimensional invariant tori  $\mathcal{T}_{I_0}^\mu \approx \mathcal{T}_{I_0}$  of the form*

$$\mathcal{T}_{I_0}^\mu = \left\{ I = I_0 + a^\mu(\psi), \quad \varphi = \psi, \quad q = Q^\mu(\psi), \quad p = P^\mu(\psi), \quad \psi \in \mathbf{T}^n \right\},$$

with  $Q^\mu(\cdot), P^\mu(\cdot), a^\mu(\cdot) = O(\mu)$ . The dynamics on  $\mathcal{T}_{I_0}^\mu$  is conjugated to the rotation of speed  $\omega$ .

In order to reduce to the previous case we want to put the tori  $\mathcal{T}_{I_0}^\mu$  “at the origin” in the  $(q, p)$  coordinates by a symplectic change of variables. As  $\mathcal{T}_{I_0}^\mu$  is isotropic, the transformation of coordinates  $(\psi, J, u, v) \rightarrow (\varphi, I, q, p)$  defined, on the covering space  $\mathbf{R}^{2(n+1)}$  of  $\mathbf{T}^n \times \mathbf{R}^n \times \mathbf{T}^1 \times \mathbf{R}^1$ , by

$$\varphi = \psi, \quad I = a^\mu(\psi) + \partial_\psi P^\mu(\psi) \cdot u - \partial_\psi Q^\mu(\psi) \cdot v + J, \quad q = Q^\mu(\psi) + u, \quad p = P^\mu(\psi) + v$$

is symplectic. In the new coordinates each invariant torus  $\mathcal{T}_{I_0}^\mu$  is simply described by  $\psi \in \mathbf{T}^n$ ,  $J = I_0$ ,  $u = v = 0$  and the new Hamiltonian writes

$$(\mathcal{K}_\mu) \quad \mathcal{K}_\mu = E_\mu + \omega \cdot J + \frac{v^2}{2} + (\cos u - 1) + P_0(\mu, u, \psi)$$

where the perturbation term is

$$P_0(\mu, u, \psi) = \left( \cos(Q^\mu + u) - \cos Q^\mu + (\sin Q^\mu)u + 1 - \cos u \right) + \mu \left( f(\psi, Q^\mu + u) - f(\psi, Q^\mu) - \partial_q f(\psi, Q^\mu)u \right)$$

and  $E_\mu$  is the energy of the perturbed invariant torus  $(a^\mu(\psi), \psi, Q^\mu(\psi), P^\mu(\psi))$ . Hamiltonian  $\mathcal{K}_\mu$  corresponds to the quasi-periodically forced pendulum

$$-\ddot{u} + \sin u = \partial_u P_0(\mu, u, \omega t + A) \quad (2.10)$$

of Lagrangian

$$L_\mu = \frac{\dot{u}^2}{2} + (1 - \cos u) - P_0(\mu, u, \omega t + A). \quad (2.11)$$

Since the Hamiltonian  $\mathcal{K}_\mu$  is not periodic in the variable  $u$  we cannot directly apply theorem 2.1 and the arguments of the previous section require some modifications. For  $\mu$  small enough there exists, near  $q_\theta$  (or more exactly its covering orbit in  $\mathbf{R}$ ), a unique *pseudo-heteroclinic* solution  $u_{A,\theta}^\mu(t)$ , which satisfies (2.10) in  $(-\infty, \theta)$  and  $(\theta, +\infty)$ , is glued with continuity at value  $u_{A,\theta}^\mu(\theta) = \pi$  and is asymptotic as  $t \rightarrow -\infty$  (resp.  $+\infty$ ) to the equilibrium 0 (resp.  $2\pi$ ). Then we define the function  $\mathcal{F}_\mu(A, \theta)$  as

$$\begin{aligned} \mathcal{F}_\mu(A, \theta) &= \int_{-\infty}^{\theta} \frac{(\dot{u}_{A,\theta}^\mu)^2}{2} + (1 - \cos u_{A,\theta}^\mu) - P_0(\mu, u_{A,\theta}^\mu, \omega t + A) dt \\ &+ \int_{\theta}^{+\infty} \frac{(\dot{u}_{A,\theta}^\mu)^2}{2} + (1 - \cos u_{A,\theta}^\mu) - P_1(\mu, u_{A,\theta}^\mu, \omega t + A) dt + 2\pi \dot{q}_A^\mu(\theta), \end{aligned}$$

where  $q_A^\mu(t) = Q^\mu(\omega t + A)$  and

$$\begin{aligned} P_1(\mu, u, \omega t + A) &= \left( \cos(q_A^\mu(t) + u) - \cos q_A^\mu(t) + \sin q_A^\mu(t) (u - 2\pi) + 1 - \cos u \right) \\ &+ \mu \left( f(\omega t + A, q_A^\mu(t) + u) - f(\omega t + A, q_A^\mu(t)) - (\partial_q f)(\omega t + A, q_A^\mu(t)) (u - 2\pi) \right). \end{aligned}$$

We define the “homoclinic function”  $\mathcal{G}_\mu(A)$  as  $\mathcal{G}_\mu(A) = \mathcal{F}_\mu(A, 0)$ .

The term  $2\pi \dot{q}_A^\mu(\theta)$  takes into account the fact that the stable and the unstable manifolds of the tori  $\mathcal{T}_{I_0}^\mu$  are *not exact* Lagrangian manifolds, see [15]. We have

**Theorem 2.3** Assume (H1) and let  $\mathcal{G}_\mu$  satisfy condition 2.1. Then the statement of theorem 2.1 holds.

For  $\mu$  small enough we have

$$\mathcal{G}_\mu(A) = \text{const} + \mu M(A) + O(\mu^2), \quad \forall A \in \mathbf{T}^n \quad (2.12)$$

where  $M(A) = \int_{-\infty}^{+\infty} [f(\omega t + A, q_0(t)) - f(\omega t + A, 0)] dt$ .

**Corollary 2.2** Assume (H1) and let  $M$  possess a non-degenerate minimum. Then, for  $\mu$  small enough, there exists a diffusion orbit with diffusion time  $T_d = O(1/\mu \log(1/\mu))$ .

### 3 Splitting of separatrices

#### 3.1 Approximation of the homoclinic function

If the frequency vector  $\omega = \omega_\varepsilon$  contains some “fast frequencies”  $\beta_i/\varepsilon^b$ ,  $b > 0$ ,  $\varepsilon$  being a small parameter, the oscillations of the Melnikov function along some directions turn out to be exponentially small with respect to  $\varepsilon$ . Hence the development (2.12) will provide a valid measure of the splitting only for  $\mu$  exponentially small with respect to  $\varepsilon$ . In order to justify the dominance of the Poincaré-Melnikov function when  $\mu = O(\varepsilon^p)$  we need more refined estimates for the error. The classical way to overcome this difficulty would be to extend analytically the function  $F_\mu(A, \theta)$  for complex values of the variables, see [15]-[10]. However it turns out that the function  $F_\mu(A, \theta)$  can *not* be easily analytically extended in a sufficiently wide complex strip (roughly speaking, the condition  $q_{A,\theta}^\mu(Re \theta) = \pi$  appearing naturally when we try to extend the definition of  $q_{A,\theta}^\mu$  to  $\theta \in \mathbf{C}$  breaks analyticity). We bypass this problem considering the action functional evaluated on different “1-bump pseudo-heteroclinic solutions”  $Q_{A,\theta}^\mu$ . This new “reduced action functional”  $\tilde{F}_\mu(A, \theta)$  has the advantage to have an analytical extension defined for  $\theta \in \mathbf{R} + i(-\pi/2, \pi/2)$ . More precisely let us assume that  $f(\varphi, q) = (1 - \cos q)f(\varphi)$  and that  $f$  can be extended to an analytical function over  $D := (\mathbf{R} + i[-a_1, a_1]) \times \dots \times (\mathbf{R} + i[-a_n, a_n])$ , for some  $a_i \geq 0$ . Then

$$f(\varphi) = \sum_{k \in \mathbf{Z}^n} f_k \exp^{ik \cdot \varphi} \quad \text{with} \quad |f_k| \leq \frac{C_s}{|k|^s} \exp\left(-\sum_{i=1}^n a_i |k_i|\right),$$

for all  $s \in \mathbf{N}$ .

Define  $\psi_0 : \mathbf{R} \rightarrow \mathbf{R}$  by  $\psi_0(t) = \cosh^2(t)/(1 + \cosh t)^3$  and set  $\psi_\theta(t) = \psi_0(t - \theta)$ . Note that  $\int_{\mathbf{R}} \psi_0(t) \dot{q}_0(t) dt \neq 0$ . By the Contraction Mapping Theorem we find near  $q_\theta$ , for  $\mu$  small enough, pseudo-heteroclinic solutions  $Q_{A,\theta}^\mu(t)$  and a constant  $\alpha_{A,\theta}^\mu$  defined by

$$-\ddot{Q}_{A,\theta}^\mu + \sin Q_{A,\theta}^\mu = \mu \sin Q_{A,\theta}^\mu f(\omega t + A) + \alpha_{A,\theta}^\mu \psi_\theta(t) \quad \text{and} \quad \int_{\mathbf{R}} (Q_{A,\theta}^\mu(t) - q_\theta(t)) \psi_\theta(t) dt = 0.$$

We define the function  $\tilde{F}_\mu : \mathbf{T}^n \times \mathbf{R} \rightarrow \mathbf{R}$  as the action functional of Lagrangian (2.3) evaluated on the “1-bump pseudo-heteroclinic solutions”  $Q_{A,\theta}^\mu(t)$ , namely

$$\tilde{F}_\mu(A, \theta) = \int_{\mathbf{R}} \mathcal{L}_\mu(Q_{A,\theta}^\mu(t), \dot{Q}_{A,\theta}^\mu(t), t) dt$$

and  $\tilde{G}_\mu : \mathbf{T}^n \rightarrow \mathbf{R}$  as  $\tilde{G}_\mu(A) = \tilde{F}_\mu(A, 0)$ . The relation between the functions  $\tilde{G}_\mu$  and  $G_\mu$  is given below

**Theorem 3.1** There exists a smooth diffeomorphism  $\psi_\mu : \mathbf{T}^n \rightarrow \mathbf{T}^n$  of the form  $\psi_\mu(A) = A + g_\mu(A)\omega$  with  $g_\mu(A) : \mathbf{T}^n \rightarrow \mathbf{R}$  satisfying  $(g_\mu(A), \partial_A g_\mu(A)) \rightarrow 0$  as  $\mu \rightarrow 0$ , such that  $G_\mu = \tilde{G}_\mu \circ \psi_\mu$ .

We now approximate the Fourier coefficients of  $\tilde{G}_\mu(A) = \sum_{k \in \mathbf{Z}^n} \tilde{G}_k \exp^{ik \cdot A}$  with the Fourier coefficients of the Poincaré-Melnikov primitive  $\Gamma(A) = \sum_{k \in \mathbf{Z}^n} \Gamma_k \exp^{ik \cdot A}$ .  $\Gamma_k$  are explicetely given by

$$\Gamma_k = f_k \frac{2\pi(k \cdot \omega)}{\sinh(k \cdot \omega \frac{\pi}{2})} \quad (3.1)$$

Set  $\|f\| = \sup_{A \in D} |f(A)|$ .

Note that the unperturbed separatrix  $q_0(t) = 4 \operatorname{arctg}(\exp(t))$  can be analytically extended up to  $|\operatorname{Im} t| < \pi/2$ . Using the Contraction Mapping Theorem, it is possible to extend analytically the function  $\tilde{F}_\mu(A, \theta)$  up to the strip  $D \times (\mathbf{R} + i(-\pi/2 + \delta, \pi/2 - \delta))$ , provided  $\mu\|f\|\delta^{-3}$  is small. By an estimate of  $\tilde{F}_\mu(A, \theta) - \mu\Gamma(A + \omega\theta)$  over its complex domain and a standard lemma on Fourier coefficients of analytical functions we obtain

**Theorem 3.2** *There is a constant  $C$  such that, for  $\mu\|f\|\delta^{-3}$  small enough,  $\forall k \neq 0, k \in \mathbf{Z}^n, \forall \delta \in (0, \frac{\pi}{2})$ ,*

$$|\tilde{G}_k - \mu\Gamma_k| \leq \frac{C\mu^2\|f\|^2}{\delta^4} \exp\left(-\sum_{i=1}^n a_i |k_i|\right) \exp\left(-|k \cdot \omega|(\frac{\pi}{2} - \delta)\right). \quad (3.2)$$

### 3.2 Three time scales

We consider three time scales Hamiltonians of the form

$$\mathcal{H} = \frac{I_1}{\sqrt{\varepsilon}} + \varepsilon^a I_2 + \frac{p^2}{2} + (\cos q - 1) + \mu(\cos q - 1)f(\varphi_1, \varphi_2), \quad I_1 \in \mathbf{R}, I_2 \in \mathbf{R}^{n-1}, \quad n \geq 2,$$

namely  $\mathcal{H}_\mu$  with  $\omega_\varepsilon = (\frac{1}{\sqrt{\varepsilon}}, \varepsilon^a)$ . Such systems have been dealt with for example in [12] and [17].

Let  $\mu\varepsilon^{-3/2}$  be small enough. We assume only that  $f$  is analytical w.r.t  $\varphi_2$  (more precisely,  $a_1 = 0$ , and for  $i \geq 2$ ,  $a_i > 0$ ,  $a_i > \pi/2$  if  $a = 0$ ). Set  $\tilde{G}_\mu(A) = \sum_{k_1 \in \mathbf{Z}} \tilde{G}_{k_1}(A_2) \exp^{ik_1 \cdot A_1}$  and  $\Gamma(A) = \sum_{k_1 \in \mathbf{Z}} \Gamma_{k_1}(A_2) \exp^{ik_1 \cdot A_1}$ . From estimate (3.2) we obtain

**Theorem 3.3** *For  $\mu\varepsilon^{-3/2}$  small there holds*

$$\begin{aligned} \tilde{G}_\mu(A_1, A_2) &= \text{Const} + \left(\mu\Gamma_0(\varepsilon, \mu, A_2) + R_0(\varepsilon, \mu, A_2)\right) + 2\operatorname{Re} \left(\mu\Gamma_1(\varepsilon, \mu, A_2) + R_1(\varepsilon, \mu, A_2)\right) e^{iA_1} \\ &+ O(\mu\varepsilon^{-1/2}\|f\| \exp^{-\frac{\pi}{\sqrt{\varepsilon}}}) \end{aligned}$$

where

$$R_0(\varepsilon, \mu, A_2) = O\left(\mu^2\|f\|^2\right) \quad \text{and} \quad R_1(\varepsilon, \mu, A_2) = O\left(\frac{\mu^2\|f\|^2}{\varepsilon^2} \exp^{-\frac{\pi}{2\sqrt{\varepsilon}}}\right).$$

**Remark 3.1** (i) *This improves the results in [17] which require  $\mu = \varepsilon^p$  with  $p > 2 + a$ .*

(ii) *Note that theorem 3.3 certainly holds in any dimension, while the results of [12], which hold for more general systems, are proved for 2 rotators only.*

(iii) *Theorem 3.3 is not in contradiction with [13].*

This theorem jointly with theorem 3.1 enables us to provide conditions implying the existence of diffusion orbits. In fact, if  $\tilde{G}_\mu$  has a proper minimum satisfying condition 2.1, so has  $G_\mu$ . For instance we obtain the following result

**Theorem 3.4** *Assume that there are  $\overline{A}_2$  and  $d, c > 0$  such that, for all small  $\varepsilon > 0$ ,  $|\Gamma_1(A_2)| > (c/\sqrt{\varepsilon})e^{-\pi/(2\sqrt{\varepsilon})}$  for all  $|A_2 - \overline{A}_2| < d$  and  $\Gamma_0(\overline{A}_2 \pm d) > \Gamma_0(\overline{A}_2) + c$ . Then, for  $\mu\varepsilon^{-3/2}$  small enough, condition 2.1 is satisfied by  $G_\mu$ , with  $\alpha = \overline{C}e^{-\pi/(2\sqrt{\varepsilon})}$ ,  $\delta = c\mu/(2\sqrt{\varepsilon})e^{-\pi/(2\sqrt{\varepsilon})}$ ,  $\overline{C}$  constant.*

**Remark 3.2** *In order to prove the splitting of the separatrices using theorem 3.3 it is necessary, according with [12] and [17], that  $\exists m, l \in \mathbf{Z}^{n-1}$  such that  $f_{0,l}, f_{1,m} \neq 0$ .*

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